

# On geodesic structures of weakly median graphs I. Decomposition and octahedral graphs

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## Abstract

We prove that the non-trivial (finite or infinite) weakly median graphs which are undecomposable with respect to gated amalgamation and Cartesian multiplication are the 5-wheels, the subhyperoctahedra different from  $K_1$ , the path  $K_{1,2}$  and the 4-cycle  $K_{2,2}$ , and the two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graphs. These prime graphs are exactly the weakly median graphs which do not have any proper gated subgraphs other than singletons. For finite graphs, these results were already proved in [H.-J. Bandelt, V.C. Chepoi, The algebra of metric betweenness I: subdirect representation, retracts, and axiomatics of weakly median graphs, preprint, 2002]. A graph  $G$  is said to have the *half-space copoint property (HSCP)* if every non-trivial half-space of the geodesic convexity of  $G$  is a copoint at each of its neighbors. It turns out that any median graph has the HSCP. We characterize the weakly median graphs having the HSCP. We prove that the class of these graphs is closed under gated amalgamation and Cartesian multiplication, and we describe the prime and the finite regular elements of this class.

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## 1. Introduction

The study of infinite graphs frequently involves topological consideration, namely the topology generated by the same subbase used to generate the convex sets (see van de Vel [25] for a topic on the weak topology). For instance, by means of the geodesic convexity, Tardif establishes in [23] a very important compactness property in median graphs. This result was extended by Chastand [8] to the class of fiber-complemented graphs with a convexity coarser than the geodesic convexity, whose convex sets are called gated sets (see [15] and Section 3 below). However, these two convexities coincide for median graphs.

Such an extension is also present in our approach, but in a different direction which is essentially based on the geodesic convexity in weakly median graphs. We will show how the close link between the two structures, convexity and topology, can be explained by the Join Hull Commutativity Property of the geodesic convexity, several consequences of which are developed in this first part.

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The class of weakly median graphs is closed under two classical operations in the context of median-like graphs (Cartesian multiplications and gated amalgamations) and a graph is *prime* if it is neither a proper Cartesian product nor a proper amalgam of non-trivial graphs. Since there is no condition of finiteness, the first main result brings some complements to a result of Bandelt and Chepoi [2] by considering the infinite case and by describing all the prime weakly median graphs.

In order to elaborate on the weak topology associated with the geodesic convexity, it is natural to examine the common subbase of both structures, which is the family of the copoints (that is, the maximal convex sets not containing a given vertex). It is easy to check that every copoint in a weakly median graph is a half-space, while the converse is not true in general, although it is so in any median graph. Thus, we investigate in Section 4 some description of the copoints of weakly median graphs; especially we introduce the half-space copoint property (HSCP) which holds if every non-trivial half-space is a copoint at each of its neighbors (median graphs are instances of such graphs). The characterization of the weakly median graphs with HSCP leads us to define the class of octahedral graphs and that of semi-octahedral graphs.

## 2. Preliminaries

The graphs we consider are undirected, without loops and multiple edges, and connected unless stated otherwise. A complete graph will be simply called a *simplex*. If  $x \in V(G)$ , the set  $N_G(x) := \{y \in V(G) : \{x, y\} \in E(G)\}$  is the *neighborhood* of  $x$  in  $G$ , and  $N_G[x] := N_G(x) \cup \{x\}$ . More generally, for a set  $X$  of vertices of a graph  $G$  we put  $N_G[X] := \bigcup_{x \in X} N_G(x)$  and  $N_G(X) := N_G[X] - X$ , and any vertex in  $N_G(X)$  will be called a *neighbor* of  $X$ . For  $A \subseteq V(G)$  we denote by  $G[A]$  the subgraph of  $G$  induced by  $A$ , and we set  $G - A := G[V(G) - A]$ . The component of  $G$  containing a vertex  $x$  is denoted by  $\mathcal{C}_G(x)$ . If  $G$  and  $H$  are two graphs, we will say that  $G$  is *H-free* if it contains no induced subgraph isomorphic to  $H$ .

A *path*  $P = \langle x_0, \dots, x_n \rangle$  is a graph with  $V(P) = \{x_0, \dots, x_n\}$ ,  $x_i \neq x_j$  if  $i \neq j$ , and  $E(P) = \{\{x_i, x_{i+1}\} : 0 \leq i < n\}$ . A *ray* or *one-way infinite path*  $\langle x_0, x_1, \dots \rangle$  and a *double ray* or *two-way infinite path*  $\langle \dots, x_{-1}, x_0, x_1, \dots \rangle$  are defined similarly. A path  $P = \langle x_0, \dots, x_n \rangle$  is called an  $(x_0, x_n)$ -*path*,  $x_0$  and  $x_n$  are its *endvertices*, while the other vertices are called its *internal* vertices,  $n = |E(P)|$  is the length of  $P$ . If  $x$  and  $y$  are vertices of a path  $P$ , then  $P[x, y]$  denotes the subpath of  $P$  joining  $x$  and  $y$ . For  $A, B \subseteq V(G)$ , an  $(A, B)$ -*path* of  $G$  is an  $(x, y)$ -path  $P$  of  $G$  such that  $V(P) \cap A = \{x\}$  and  $V(P) \cap B = \{y\}$ ; and an  $(A, B)$ -*linkage* of  $G$  is a set of pairwise disjoint  $(A, B)$ -paths of  $G$ . If there exists an infinite  $(A, B)$ -linkage in  $G$ , then we say that  $A$  and  $B$  are *infinitely linked* in  $G$ . By Menger's theorem,  $A$  and  $B$  are infinitely linked in  $G$  if and only if they cannot be separated in  $G$  by removing finitely many vertices.

The usual *distance* in a graph  $G$  between two vertices  $x$  and  $y$ , that is the length of an  $(x, y)$ -*geodesic* (i.e., shortest  $(x, y)$ -path) in  $G$ , is denoted by  $d_G(x, y)$ . A subgraph  $H$  of  $G$  is *isometric* in  $G$  if  $d_H(x, y) = d_G(x, y)$  for all vertices  $x$  and  $y$  of  $H$ . The *diameter* of  $G$  is  $\text{diam}(G) := \sup\{\text{dist}_G(x, y) : x, y \in V(G)\}$ . If  $x$  is a vertex of  $G$  and  $r$  a non-negative integer, the set  $B_G(x, r) := \{y \in V(G) : d_G(x, y) \leq r\}$  is the *ball of center  $x$  and radius  $r$  in  $G$* , and the set  $S_G(x, r) := \{y \in V(G) : d_G(x, y) = r\}$  is the *sphere of center  $x$  and radius  $r$  in  $G$* . The *interval*  $I_G(x, y)$  of two vertices  $x$  and  $y$  of a graph  $G$  is the set of vertices of all  $(x, y)$ -geodesics in  $G$ .

The *Cartesian product* of a family of graphs  $(G_i)_{i \in I}$  is the graph denoted by  $\square_{i \in I} G_i$  (or simply by  $G_1 \square G_2$  if  $|I| = 2$ ) with  $\prod_{i \in I} V(G_i)$  as vertex set and such that, for every vertices  $u$  and  $v$ ,  $\{u, v\}$  is an edge whenever there exists a unique  $j \in I$  with  $\{\text{pr}_j(u), \text{pr}_j(v)\} \in E(G_j)$  and  $\text{pr}_i(u) = \text{pr}_i(v)$  for every  $i \in I - \{j\}$  (where  $\text{pr}_i$  is the  $i$ th *projection* of  $\prod_{i \in I} V(G_i)$  onto  $V(G_i)$ ). Connected components of a Cartesian product of connected graphs are called *weak Cartesian products* (see [18]). Clearly, the Cartesian product coincides with the weak Cartesian product provided that  $I$  is finite and the factors are connected.

If  $G$  and  $H$  are two graphs, a map  $f : V(G) \rightarrow V(H)$  is a *contraction* if  $f$  preserves or contracts the edges, i.e., if  $f(x) = f(y)$  or  $\{f(x), f(y)\} \in E(H)$  whenever  $\{x, y\} \in E(G)$ . In particular, the projections of a Cartesian product are contractions. A contraction  $f$  from  $G$  onto an induced subgraph  $H$  of  $G$  is a *retraction*, and  $H$  is a *retract* of  $G$ , if its restriction to  $V(H)$  is the identity.

We recall that a graph  $G$  is *weakly modular* if it satisfies the following two conditions:

*Triangle condition:* For any three vertices  $x_0, x_1, x_2$  with  $1 = d_G(x_1, x_2) < d_G(x_0, x_1) = d_G(x_0, x_2)$ , there exists a common neighbor  $u$  of  $x_1$  and  $x_2$  such that  $d_G(x_0, u) = d_G(x_0, x_1) - 1$ .

*Quadrangle condition:* For any four vertices  $x_0, x_1, x_2, x_3$  with  $d_G(x_1, x_3) = d_G(x_2, x_3) = 1$  and  $d_G(x_0, x_1) = d_G(x_0, x_2) = d_G(x_0, x_3) - 1$ , there exists a common neighbor  $u$  of  $x_1$  and  $x_2$  such that  $d_G(x_0, u) = d_G(x_0, x_1) - 1$ .

In this paper we will consider the median-like weakly modular graphs. A *quasi-median* of a triple  $(u_0, u_1, u_2)$  of vertices of a graph  $G$  is a triple  $(x_0, x_1, x_2)$  of vertices of  $G$  such that:  $\{x_i, x_j\} \subseteq I_G(u_i, u_j)$  for all  $i, j \in \{0, 1, 2\}$  with  $i \neq j$  and  $d_G(x_0, x_1) = d_G(x_1, x_2) = d_G(x_2, x_0) = k$  where  $k$  is minimal with respect to these conditions. This non-negative integer  $k$  is called the *size* of the quasi-median. A quasi-median of size 1 is called a *pseudo-median*, and a quasi-median of size 0 is reduced to a single vertex which is called the *median* of the triple  $(u_0, u_1, u_2)$ . Due to the minimality of its size a quasi-median  $(x_0, x_1, x_2)$  forms a particular *metric triangle* of  $G$ , which means that  $I_G(x_i, x_j) \cap I_G(x_i, x_k) = \{x_i\}$  for every triple  $(i, j, k)$  of pairwise disjoint elements of  $\{0, 1, 2\}$ . According to Chepoi [11], a graph  $G$  is weakly modular if and only if, for every metric triangle  $(x_0, x_1, x_2)$  and every triple  $(i, j, k)$  of pairwise disjoint elements of  $\{0, 1, 2\}$ , all vertices in the interval  $I_G(x_j, x_k)$  are at the same distance  $d_G(x_i, x_j)$  from  $x_i$ .

In a weakly modular graph every triple of vertices has a quasi-median. In particular, a *weakly median* graph is a weakly modular graph in which every triple of vertices has a unique quasi-median, or equivalently a weakly modular graph that does not contain any two vertices with an unconnected triple of common neighbors. A *pseudo-median* (resp. *median*) graph is a weakly median graph in which every triple of vertices has a quasi-median of size 0 or 1 (resp. 0). A subgraph  $H$  of a weakly median (resp. pseudo-median, median) graph  $G$  is called a *weakly median subgraph* (resp. *pseudo-median subgraph*, *median subgraph*) of  $G$  if, for every triple of vertices of  $H$ , their weakly median in  $G$  belongs to  $H$ .

A set  $A$  of vertices of a graph  $G$  is *geodesically convex* in  $G$ , for short *convex*, if it contains the interval  $I_G(x, y)$  for all  $x, y \in A$ . We will say that a subgraph of a graph  $G$  is convex if its vertex set is convex in  $G$ . The set of all convex subsets of  $V(G)$  is the *geodesic convexity* on  $G$ . The *convex hull*  $co_G(A)$  of a set  $A$  of vertices of  $G$  is the smallest convex set of  $G$  containing  $A$ . The convex hull of a finite set is called a *polytope*. A *copoint* at a vertex  $x$  is a convex set  $C$  which is maximal with the property  $x \notin C$ . Each convex set is clearly the intersection of a family of copoints, that is the set of all copoints is an intersectional subbase of the convexity structure. A subset  $H$  of  $V(G)$  is a *half-space* if  $H$  and  $V(G) - H$  are convex. See van de Vel [25] for a detailed study of abstract convex structures.

We will now recall some properties of the geodesic convexity of weakly median graphs.

**Proposition 2.1** (Chepoi [12]). *Every interval of a weakly median graph is convex.*

We recall that an abstract convex structure  $(X, \mathcal{C})$  is JHC (Join-hull commutativity) if, for any convex set  $C \subseteq X$  and any  $u \in X$ , the convex hull of  $\{u\} \cup C$  equals the union of the convex hull of  $\{u, v\}$  for all  $v \in C$ .

**Proposition 2.2** (Chepoi [12]). *The geodesic structure of a weakly median graph is JHC.*

It follows by the two preceding lemmas that:

**Corollary 2.3.** *Let  $G$  be a weakly median graph,  $C$  a convex subset of  $V(G)$  and  $u \in V(G) - C$ . Then  $co_G(\{u\} \cup C) = \bigcup_{c \in C} I_G(u, c)$ .*

**Corollary 2.4** (Polat [22, Corollary 5.7]). *The polytopes of an interval-finite weakly median graph are the finite convex sets.*

As a consequence of a more general result of Chepoi [13, Theorem 11], we have the following property:

**Proposition 2.5.** *The geodesic convexity of a weakly median graph is  $S_4$ , i.e., if  $C, D \subseteq V(G)$  are disjoint convex sets, then there is a half-space  $H$  with  $C \subseteq H$  and  $D \in V(G) - H$ .*

### 3. Elementary weakly median graphs

An induced subgraph  $H$  of a graph  $G$  is called a *gated subgraph* of  $G$  if, for every  $x$  in  $V(G)$ , there exists a vertex  $y$  (the *gate* of  $x$ ) in  $H$  such that  $y \in I_G(x, z)$  for every  $z \in V(H)$  (see [15]). Obviously, every gated subgraph of a weakly median graph is a convex subgraph. Conversely, the following lemma characterizes convex subgraphs which are gated subgraphs (a subgraph  $H$  is  $\Delta$ -closed if, for every triangle having two vertices in  $V(H)$ , the third vertex belongs to

$V(H)$  as well). The  $\Delta$ -closure (resp. gated hull) of a subgraph (or a set of vertices)  $S$  of  $G$  is the smallest  $\Delta$ -closed induced (resp. gated) subgraph of  $G$  containing  $S$ .

**Lemma 3.1** (Bandelt and Chepoi [2, Lemma 1]). *A convex subgraph of a weakly median graph is gated if and only if it is  $\Delta$ -closed.*

The study of gated subgraphs is particularly suitable in the context of median-like classes of graphs, in conjunction with two operations: the Cartesian multiplications and the gated amalgamations. Following Mulder [20], a graph  $G$  is the *gated amalgam* of two graphs  $G_1$  and  $G_2$  if  $G_1$  and  $G_2$  are isomorphic to two intersecting gated subgraphs of  $G$  whose union is  $G$ .

A graph with at least two vertices is called a *prime* graph if it is neither a proper Cartesian product nor a proper gated amalgam of non-trivial graphs (i.e., distinct from singletons). Furthermore a graph with at least two vertices is called an *elementary* graph (see [8]) if it does not have any proper gated subgraphs other than singletons. As a consequence of a result of Chastand [7], a weakly median graph is elementary if and only if it is the  $\Delta$ -closure of any of its edges.

We recall that a  $n$ -wheel ( $n \geq 3$ ) is a cycle of length  $n$  with a “central” vertex adjacent to all vertices of the cycle, that a *hyperoctahedron* is a (finite or infinite) simplex of cardinality at least 6 minus a perfect matching (i.e., a multipartite graph of the form  $K_{2,2,2,\dots}$ ), that a *subhyperoctahedron* is an induced subgraph of a hyperoctahedron (i.e., a multipartite graph of the form  $K_{i_1,i_2,\dots}$  with  $1 \leq i_j \leq 2$ ), and finally that a graph is *bridged* if it contains no isometric cycle of length greater than 3. Note that bridged graphs are weakly modular. Moreover, as a consequence of a result of Chepoi [11, Theorem 7], one can easily prove that the  $\Delta$ -closure of an edge of a bridged graph  $G$  is a gated subgraph of  $G$ . In [16] Farber and Jamison gave a useful property of a bridged graph.

**Lemma 3.2** (Farber and Jamison [16, Theorem 6.4]). *If  $x$  is not a cutvertex of a bridged graph  $G$ , then  $G[N_G(x)]$  is connected.*

**Proposition 3.3.** *A two-connected bridged graph is an elementary graph which is the  $\Delta$ -closure of any of its edges.*

**Proof.** Let  $G$  be a two-connected bridged graph, and let  $X$  be the  $\Delta$ -closure of one of its edges. Suppose that it is a proper subgraph of  $G$ . Since  $G$  is connected there are two vertices  $x \in V(X)$  and  $y \in V(G - X)$  which are adjacent. Because  $X$  is the  $\Delta$ -closure of an edge, it follows that  $x$  is adjacent to another vertex  $z$  of  $X$ . Moreover  $x$  is not a cutvertex since  $G$  is two-connected. Hence, by Lemma 3.2,  $N_G(x)$  induces a connected subgraph of  $G$ . Therefore, every neighbor of  $X$ , and in particular  $y$ , belongs to the  $\Delta$ -closure of the edge  $\{x, z\}$ , contrary to the fact that  $y \notin V(X)$ . Consequently  $X = G$ .  $\square$

Bandelt and Chepoi established in [2] a characterization of the finite prime weakly median graphs. We will now extend this result by characterizing all prime weakly median graphs.

**Theorem 3.4.** *A non-trivial weakly median graph is prime if and only if it is elementary. These graphs are: the 5-wheels, the subhyperoctahedra different from  $K_1$ , the path  $K_{1,2}$  and the 4-cycle  $K_{2,2}$ , and the two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graphs. The latter bridged graphs are exactly the graphs which can be realized as two-connected countable plane graphs such that all inner faces are triangles, all inner vertices have degrees greater than 5, and at most finitely many vertices lie on the interior of each region of the plane bounded by a cycle.*

**Proof.** The proof will be partly similar to that of [2, Theorem 1] and we will need to refer to several auxiliary lemmas of these authors.

(1) Suppose that a weakly median graph  $G$  is not prime. If it is a gated amalgam of non-trivial weakly median graphs, then it is clearly not elementary. If  $G = G_1 \square G_2$  where  $G_1$  and  $G_2$  are non-trivial weakly median graphs, then, for any  $x \in V(G_2)$ , the graph  $G_1 \square \{x\}$  is convex and  $\Delta$ -closed, and thus is a proper non-trivial gated subgraph of  $G$ , which proves that  $G$  is not elementary. Therefore, an elementary weakly median graph is prime. The converse was proved by Bandelt and Chepoi [3, Lemma 6].

(2a) We will now prove that all graphs listed in the theorem are elementary. This is clear for 5-wheels by [2, Lemma 3] and for two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graphs by Proposition 3.3. Now if a subhyperoctahedron is different

from  $K_{1,2}$  and from  $K_{2,2}$ , then it contains an induced 4-wheel  $K_{1,2,2}$  or it is a simplex. Hence it is elementary either by [2, Lemma 3] or by the fact that a simplex is clearly elementary since a gated subgraph is  $\Delta$ -closed.

(2b) Conversely assume that  $G$  is neither a singleton nor any of the elementary graphs listed in Theorem 3.4. Then, by [2, Lemma 6],  $G$  is not a two-connected bridged graph. We have to show that  $G$  has a proper gated subgraph with at least two vertices. If  $G$  includes an induced 4- or 5-wheel, then, by [2, Lemmas 2 and 3], it has a proper gated subgraph which is a subhyperoctahedron or a 5-wheel. Suppose that  $G$  contains no induced 4- or 5-wheels. If  $G$  still contains some triangle, then, by [2, Lemma 5], it has a proper gated subgraph which is bridged and two-connected. If  $G$  contains no triangle, then, by [2, Lemmas 2 and 4], there are no odd cycles at all, whence  $G$  is a median graph, and thus any edge of  $G$  induces a proper gated subgraph. Therefore in every case,  $G$  is not elementary.

(3) We will now prove the last part of Theorem 3.4. Let  $G$  be a two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graph.

(3a) We will first briefly recall some concepts of infinite combinatorics by limiting ourselves to the case of the ordinal  $\omega_1$ . A *club* in  $\omega_1$  is a closed unbounded subset  $X$  of  $\omega_1$ , that is, for every  $\alpha < \omega_1$ ,  $X - \alpha \neq \emptyset$  and  $\sup(X \cap \alpha) \in X$  whenever  $X \cap \alpha \neq \emptyset$ . A *stationary* subset of  $\omega_1$  is a set  $S$  which has a non-empty intersection with every club in  $\omega_1$ . A function  $f : A \rightarrow \omega_1$  where  $A \subseteq \omega_1$  is *regressive* if  $f(\alpha) < \alpha$  for every  $\alpha \in A$ . By a theorem of Fodor [17], if  $f$  is a regressive function on a stationary subset  $S$  of  $\omega_1$ , then  $f^{-1}(\alpha)$  is stationary for some  $\alpha < \omega_1$ .

(3b) We will show that  $G$  is countable. Suppose that  $G$  is uncountable. We will construct a sequence  $(x_\alpha)_{\alpha < \omega_1}$  of vertices of  $G$  as follows. Let  $x_0, x_1$  be two adjacent vertices of  $G$ . By Proposition 3.3,  $G$  is the  $\Delta$ -closure of the edge  $\{x_0, x_1\}$ . Let  $\alpha$  be an ordinal such that  $1 \leq \alpha < \omega_1$ . Suppose that  $x_\beta$  has already been constructed for every  $\beta < \alpha$  in such a way that  $G_\alpha := G[\{x_\beta : \beta < \alpha\}]$  is an induced connected subgraph of  $G$ . Since  $G$  is the  $\Delta$ -closure of  $\{x_0, x_1\}$  and because  $G \neq G_\alpha$ , there is a common neighbor  $x \in V(G - G_\alpha)$  of two adjacent vertices of  $G_\alpha$ . Put  $x_\alpha := x$ .

By the construction there are two functions  $f, g : \omega_1 \rightarrow \omega_1$  such that, for every  $\alpha < \omega_1$ ,  $f(\alpha) < g(\alpha) < \alpha$  and the vertices  $x_\alpha, x_{f(\alpha)}$  and  $x_{g(\alpha)}$  are pairwise adjacent. Then  $f$  is regressive, and thus, by Fodor's theorem, there exists  $\gamma < \omega_1$  such that  $f^{-1}(\gamma)$  is stationary. Now the restriction of  $g$  to  $f^{-1}(\gamma)$  is also regressive, and thus, by Fodor's theorem, there exists  $\delta < \omega_1$  such that  $g^{-1}(\delta)$  is a stationary (hence infinite) subset of  $f^{-1}(\gamma)$ . Therefore, for every  $\alpha \in g^{-1}(\delta)$ , the vertices  $x_\alpha, x_\gamma, x_\delta$  are pairwise adjacent, contrary to the fact that  $G$  is  $K_4$ - and  $K_{1,1,3}$ -free.

(3c) We will show that  $G$  can be realized as a two-connected countable plane graph such that all inner faces are triangles, all inner vertices have a degree greater than 5 and at most finitely many vertices lie on the interior of each region of the plane bounded by a cycle. We are done if  $G$  is finite by [2, Theorem 1]. Assume that  $G$  is infinite, and thus countably infinite by (3b).

**Claim 1.** *The neighborhood of any vertex of  $G$  induces either a (finite or infinite) path or a cycle of length greater than 5.*

Let  $x \in V(G)$ . Since  $G$  is two-connected,  $x$  is not a cutvertex. Hence,  $G[N_G(x)]$  is connected by Lemma 3.2. The claim follows from the fact that two adjacent vertices of  $G$  have at most two common neighbors since  $G$  is  $K_4$ - and  $K_{1,1,3}$ -free.

Now, we will use a characterization of bridged graphs by breadth-first search (BFS). We first recall what we call a BFS-order. Let  $G$  be a connected graph. A well-order  $\leq$  on  $V(G)$  is called a *BFS-order* if there exists a family  $(A_x)_{x \in V(G)}$  of subsets of  $V(G)$  such that, for every  $x \in V(G)$ :

- (i)  $x \in A_x$ ;
- (ii) if  $x \leq y$ , then  $A_x$  is an initial segment of  $A_y$  with respect to the induced order;
- (iii)  $A_x = A_{(x)} \cup N_G(x)$  where  $A_{(x)} := \{x\}$  if  $x$  is the least element of  $(V(G), \leq)$ , and otherwise  $A_{(x)} := \bigcup_{y < x} A_y$ .

The vertex  $x$  will be called the *father* of each element of  $A_x - A_{(x)}$ . We will denote by  $\phi$ , and call *father function*, the self-map of  $V(G)$  such that  $\phi(u) = u$  if  $u$  is the smallest element of  $(V(G), \leq)$ , and  $\phi(x)$  is the father of  $x$  for every  $x \in V(G - u)$ .

We will also say that a vertex  $x$  of a graph  $G$  is *dominated* by another vertex  $y$  in  $G$  if  $N_G[x] \subseteq N_G[y]$ . In [10], Chastand et al. proved that a connected graph  $G$  is bridged if and only if the father function of any BFS-order  $\leq$  on  $V(G)$  is such that every vertex  $x$  of  $G$  is dominated by its father in the subgraph of  $G$  induced by the set  $\{y \in V(G) : y \leq x\}$ .



We will construct a planar embedding  $\sigma$  of  $G$  such that the drawing  $\sigma(G)$  of  $G$  has the required properties. Let  $\leq$  be a BFS-order on  $V(G)$ , and let  $\phi$  be its father function. Let  $(x_n)_{n < \omega}$  be the sequence of vertices of  $G$  defined as follows:

- $x_0$  is the smallest element of  $(V(G), \leq)$
- $x_{n+1} := \min\{z \in V(G) : z > y \text{ for every } y \in \phi^{-1}(\phi(x_n))\}$ .

For  $n \geq 1$  put  $G_n := G[\{x \in V(G) : x < x_n\}]$ . We will construct a sequence  $\sigma_1, \sigma_2, \dots$  such that, for every positive integer  $n$ ,  $\sigma_n$  is a planar embedding of  $G_n$  with the following properties:

- $\sigma_n \subset \sigma_{n+1}$ , i.e.,  $\sigma_n$  is the restriction of  $\sigma_{n+1}$  to  $V(G_n)$ ,
- $\sigma_n(G_n)$  is a two-connected countable plane graph such that all inner faces are triangles, all inner vertices have a degree greater than 5, there are at most finitely many vertices lying on the interior of each region of the plane bounded by a cycle, and for each  $x < x_n$ , if  $x_n \leq y$  for some  $y \in \phi^{-1}(x)$ , then  $\sigma_n(x)$  lies on the boundary of an outer face of  $\sigma_n(G_n)$ .

The construction of  $\sigma_1$  is obvious. Suppose that  $\sigma_n$  has already been constructed for some  $n \geq 1$ . Note that, since  $n \geq 1$ ,  $x_0 \leq \phi^2(x_n) \in N_G(\phi(x_n))$ .

**Claim 2.** Let  $y \in N_G(\phi(x_n))$  with  $x_n \leq y$ . If  $y$  is adjacent to some  $z < x_n$  with  $z \neq \phi(x_n)$ , then  $z \in N_G(\phi(x_n))$ .

This is clear because  $z \in V(G_n)$  and  $\phi(x_n) = \phi(y)$  dominates  $y$  in  $G[\{x \in V(G) : x \leq y\}]$  by [10, Theorem 4.3].

**Claim 3.** Let  $\langle y_0, y_1, y_2 \rangle$  be a subpath of  $G[N_G(\phi(x_n))]$  with  $x_n \leq y_0$  and  $x_n \leq y_2$ . Then  $x_n \leq y_1$ .

Suppose that  $y_1 < x_n$ . We distinguish two cases. If  $d_G(x_0, y_1) = d_G(x_0, x_n)$ , then  $\phi(y_1) < \phi(x_n)$ . Since  $G$  is a bridged graph, the ball  $B_G(x_0, d_G(x_0, \phi(x_n)))$  is convex, and thus  $\phi(x_n)$  and  $\phi(y_1)$  must be adjacent because  $y_1$  is a common neighbor of these two vertices which does not belong to the preceding ball. If  $d_G(x_0, y_1) = d_G(x_0, \phi(x_n))$ , then  $\phi(x_n) < y_1$ . Hence  $\phi(x_n)$  and  $\phi(y_1)$  are adjacent because  $\phi(y_1)$  dominates  $y_1$  in  $G[\{x \in V(G) : x \leq y_1\}]$  by [10, Theorem 4.3]. Therefore, in both cases, the fact that  $\phi(x_n)$  and  $\phi(y_1)$  are adjacent implies that  $N_G(\phi(x_n))$  does not induce a path or a cycle, contrary to Claim 1.

**Claim 4.** There exists  $y \in N_G(\phi(x_n))$  such that  $y < x_n$ .

This is due to the fact that  $\phi(x_n)$  is not a cutvertex because  $G$  is two-connected by hypothesis.

It follows from all these claims that the set  $\phi^{-1}(\phi(x_n))$  induces a path or a ray.

Then we can easily extend  $\sigma_n$  to a planar embedding  $\sigma_{n+1}$  of  $G_{n+1}$  such that, for every  $y \in \phi^{-1}(\phi(x_n))$ ,  $\sigma_{n+1}(y)$  lies on the outer face of  $\sigma_n(G_n)$  whose boundary contains  $\sigma_n(\phi(x_n))$ , which is possible by the induction hypothesis, and in such a way that  $\sigma_{n+1}(y)$  lies on the boundary of an outer face of  $\sigma_{n+1}(G_{n+1})$ .

Clearly, by the construction and by the induction hypothesis, all inner faces of the plane graph  $\sigma_{n+1}(G_{n+1})$  are triangles and, for each  $x < x_{n+1}$ , if  $x_{n+1} \leq y$  for some  $y \in \phi^{-1}(x)$ , then  $\sigma_{n+1}(x)$  lies on the boundary of an outer face of  $\sigma_{n+1}(G_{n+1})$ . Now, if  $\sigma_{n+1}(G_{n+1})$  has an inner vertex which is not an inner vertex of  $\sigma_n(G_n)$ , then this vertex is  $x_n$ , and thus  $G[N_G(x_n)]$  is a cycle which is of length greater than 5 because  $G$  is bridged. Hence,  $x_n$  has a degree greater than 5. Finally, let  $C = \langle y_0, \dots, y_k, y_0 \rangle$  be a cycle of  $\sigma_{n+1}(G_{n+1})$  which is not a cycle of  $\sigma_n(G_n)$ . Then there is at least one vertex of  $C$  which belongs to  $\phi^{-1}(\phi(x_n))$ . Suppose that  $y_0 \in \phi^{-1}(\phi(x_n))$ , and let  $i \leq k$  be the greatest integer such that  $y_i \in \phi^{-1}(\phi(x_n))$ . Since  $\phi(x_n)$  dominates  $y_0$  and  $y_i$  in  $\sigma_{n+1}(G_{n+1})$ , it follows that  $y_1, y_{i-1}, y_{i+1}, y_k \in N_G[\phi(x_n)]$  (with  $y_{i-1} := y_k$  if  $i = 0$ ). Because  $\phi^{-1}(\phi(x_n))$  induces a path or a ray, there is an induced  $\langle y_0, y_i \rangle$ -path  $P$  whose vertices belongs to  $\phi^{-1}(\phi(x_n))$ . This path  $P$  is or is not equal to  $\langle y_0, \dots, y_i \rangle$ . If  $P \neq \langle y_0, \dots, y_i \rangle$  and if  $j$  is the smallest integer such that  $0 < j < i$  and  $y_j \notin V(P)$ , then  $y_j = \phi(x_n)$ . Moreover, because  $\sigma_{n+1}(G_{n+1})$  is a plane graph by construction, it follows that  $y_k \in V(P)$  for every  $k \neq j$  with  $0 \leq k \leq i$ , and then  $\langle y_0, \dots, y_i \rangle = P[y_0, y_{j-1}] \cup \langle y_{j-1}, \phi(x_n), y_{j+1} \rangle \cup P[y_{j+1}, y_i]$ . The cycle  $C' := P \cup \langle y_i, \dots, y_k, y_0 \rangle$ , which can be equal to  $C$  if  $i = 0$ , i.e., if  $y_0$  is the only vertex of  $C$  in  $\phi^{-1}(\phi(x_n))$ , is such that the interior of the region of the plane bounded by  $C'$  contains the one bounded by  $C$ . The path  $\langle y_{i+1}, \dots, y_k \rangle$  (with  $y_{i+1} := y_0$  if  $i = k$ ) together with the two edges  $\{\phi(x_n), y_{i+1}\}$  and  $\{\phi(x_n), y_k\}$  form a cycle  $C''$ . Clearly the vertices lying on the interior of the region of the plane bounded by  $C'$  are the vertices which lie on the interior of the region of the plane bounded by  $C''$  plus possibly the vertex  $\phi(x_n)$ . From the induction hypothesis, it follows that there are only finitely many such vertices.

Then  $\sigma := \bigcup_{n \geq 1} \sigma_n$  is a planar embedding of  $G$  which has the required properties.

(3d) Conversely let  $G$  be a two-connected countable plane graph such that all inner faces are triangles, all inner vertices have a degree greater than 5 and at most finitely many vertices lie on the interior of each region of the plane bounded by a cycle. Consider any triangle of  $G$ . Together with its interior in the plane it constitutes a finite plane graph  $H$  to which [2, Lemma 7] applies. Following Bandelt and Chepoi in the proof of [2, Theorem 1], we infer that each vertex of the boundary triangle must have degree 2 in  $H$ , that is,  $H$  includes no inner vertex. Hence, all triangles of  $G$  constitute inner faces (and vice versa). In particular,  $G$  is  $K_4$ - and  $K_{1,1,3}$ -free. It remains to show that  $G$  is bridged. Let  $C$  be a cycle of  $G$  of length greater than 3. Then  $C$  together with its interior in the plane constitutes a finite plane graph  $H$  such that all inner faces are inner faces of  $G$  and thus are triangles, and all inner vertices have the same degrees in  $H$  as in  $G$  and thus have a degree greater than 5. Then, by [2, Theorem 1],  $H$  is a bridged graph. Hence  $C$  is not isometric in  $H$ , and a fortiori not in  $G$ .  $\square$

Theorem 3.4 will enable us to confirm the correctness of two results of Chastand [9, Proposition 2.2.1 and Corollary 3.4.1.4] which, in error, were stated for any (finite or infinite) weakly median graph while they were proved only for finite ones.

A self-contraction  $f$  of a graph  $G$  is a *mooring* onto a vertex  $u$  of  $G$  if  $f(u) = u$  and  $\{x, f(x)\}$  is an edge of  $G[I_G(x, u)]$  for every vertex  $x \neq u$ ; a graph is *moorable* if, for every vertex  $u \in V(G)$ , there exists a mooring of  $G$  onto  $u$  (or, equivalently,  $G$  has a geodesic 1-combing with respect to all base points, as defined by Chepoi in [14]). Now, by [10, Proposition 4.4] every bridged graph is moorable, and it is straightforward to check that a 5-wheel and every subhyperoctahedron are moorable since they are graphs with diameter 2. Thus, every elementary weakly median graph is moorable.

Consequently, by Theorems 3.2.1 and 3.3.1 in Chastand [9], every (finite or infinite) weakly median graph is a retract of a (weak) Cartesian product of elementary weakly median graphs. By [8, Theorem 6.1, Corollary 6.2], every factor of this Cartesian product is isomorphic to some elementary gated subgraph of  $G$ , as a representative of a parallelism relation between the elementary gated subgraphs.

**Theorem 3.5.** *Every weakly median graph  $G$  is a retract of a Cartesian product of elementary weakly median graphs and each factor of this product is isomorphic to some elementary gated subgraph of  $G$ .*

#### 4. Octahedral weakly median graphs

Because every singleton is convex, it follows from Proposition 2.5 that *every copoint in a weakly median graph is a half-space*. If the converse is true for median graphs, it is generally not so for any weakly median graph. In this section we will study the weakly median graphs for which half-spaces are copoints. First we will give a characterization of the half-spaces in a weakly median graph which are copoints.

For  $A \subseteq V(G)$  and  $x \in V(G) - A$  we define the *threshold of  $A$  for  $x$*  as the set

$$A^{(x)} := \{a \in A : d_G(a, x) = d_G(A, x)\}.$$

**Lemma 4.1.** *Let  $A$  be a convex set of vertices of a weakly median graph  $G$ , and  $x \in V(G) - A$ . Then:*

- (i)  $A^{(x)}$  is convex.
- (ii)  $I_G(a, x) \cap A^{(x)} \neq \emptyset$  for every  $a \in A$ .

**Proof.** (i) Let  $a, b \in A^{(x)}$ . By the definition of  $A^{(x)}$  and the fact that  $A$  is convex, it follows that the quasi-median of  $(x, a, b)$  is of the form  $(x', a, b)$ . Hence, because a quasi-median is a metric triangle, it follows that  $d_G(x, y) = d_G(x, a) = d_G(x, A)$  for every  $y \in I_G(a, b)$ , which proves that  $I_G(a, b) \subseteq A^{(x)}$ .

(ii) Let  $a \in A$  and  $b \in A^{(x)}$ . We are done if  $b \in I_G(a, x)$ . Suppose that  $b \notin I_G(a, x)$ . Then the size of the quasi-median  $(x', a', b')$  of  $(x, a, b)$  is at least 1. By the convexity of  $A$ ,  $a'$  and  $b'$  belongs to  $A$ , and then  $b' = b$  because  $b \in A^{(x)}$ . Therefore, since  $d_G(x, a') = d_G(x, b)$ , it follows that  $a' \in A^{(x)} \cap I_G(a, x)$ .  $\square$

In particular, if  $H$  is a gated subgraph of a graph  $G$ , then the threshold of  $V(H)$  for any vertex of  $G$  is a singleton.

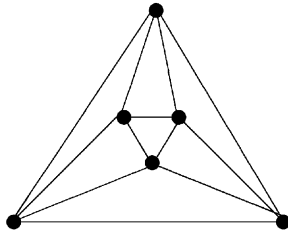


Fig. 1.

**Lemma 4.2.** Let  $G$  be a weakly median graph,  $A$  a convex subset of  $V(G)$  and  $u \in N_G(A)$ . Then  $co_G(A \cup \{u\}) \subseteq N_G[A]$ .

**Proof.** By Corollary 2.3,  $co_G(A \cup \{u\}) = \bigcup_{a \in A} I_G(u, a)$ . Let  $a \in A$  and let  $\langle x_0, \dots, x_n \rangle$  be a  $(u, a)$ -geodesic with  $x_0 = u$  and  $x_n = a$ . Suppose that  $x_j \notin N_G[A]$  for some  $j$  with  $0 \leq j \leq n$ . Then there exists  $i$  with  $0 \leq i < n$  such that  $x_i \in N_G(A)$  and  $d_G(x_{i+1}, A) = 2$ . By Lemma 4.1,  $I_G(x_i, a) \cap A^{(x_i)} \neq \emptyset$ . Let  $b \in I_G(x_i, a) \cap A^{(x_i)}$ . Then  $d_G(a, b) = d_G(a, x_i) - 1$ . Therefore, by the quadrangle condition, there is a vertex  $y \in N_G(b) \cap N_G(x_{i+1}) \cap B_G(a, d_G(a, x_i) - 2)$ . By the convexity of  $A$ , the vertex  $y \in A$ , which implies that  $d_G(x_{i+1}, A) = 1$ , contrary to the choice of  $x_{i+1}$ .  $\square$

**Proposition 4.3.** A half-space  $A$  in a weakly median graph  $G$  is a copoint at a vertex  $u$  if and only if  $u \in N_G(A)$  and  $A^{(u)} - A^{(x)} \neq \emptyset$  for every  $x \in N_G(u) \cap N_G(A)$ .

**Proof.** (a) Suppose that  $A$  is a copoint at a vertex  $u$ . We are done if  $N_G(A) = \{u\}$ . Suppose that  $N_G(A) \neq \{u\}$ . Since  $A$  is a maximal convex set which does not contain  $u$ , it follows that  $u \in co_G(A \cup \{x\})$  for any vertex  $x \in V(G) - (A \cup \{u\})$ , and in particular for any  $x \in N_G(A) - \{u\}$ . By Lemma 4.2, this implies that  $u \in N_G(A)$ . Let  $x \in N_G(u) \cap N_G(A)$ . By Corollary 2.3, there is a  $y \in A$  such that  $u \in I_G(x, y)$ . Then, by Lemma 4.1(ii), there is a  $y' \in A \cap N_G(u) \cap I_G(u, y)$ . Hence  $u \in I_G(x, y')$ , and thus  $y' \notin A^{(x)}$ .

(b) Conversely, suppose that  $u \in N_G(A)$  and  $A^{(u)} - A^{(x)} \neq \emptyset$  for every  $x \in N_G(u) \cap N_G(A)$ . Assume that  $A$  is not a copoint at  $u$ . Then there exists a vertex  $b \in N_G(A) - \{u\}$  such that  $u \notin co_G(A \cup \{b\}) =: B$ . By Lemma 4.1(ii),  $I_G(u, b) \cap B^{(u)} \neq \emptyset$ . Hence, since  $I_G(u, b) \cap A = \emptyset$  by the convexity of  $V(G) - A$ , it follows that  $I_G(u, b) \cap B^{(u)} \subseteq V(G) - A$ . Let  $x \in I_G(u, b) \cap B^{(u)}$ . Then  $u \notin co_G(A \cup \{x\})$ . This implies that  $x$  is adjacent to every element of  $A^{(u)}$ , contrary to the hypothesis.  $\square$

By the preceding two results, a convex set  $C$  in a weakly median graph  $G$  is a copoint at each of its neighbors if and only if  $co_G(C \cup \{x\}) = N_G[C]$  for every  $x \in N_G(C)$ . In fact, if  $C$  has more than one neighbor, then it suffices that this property is satisfied for two distinct neighbor of  $C$ .

We will say that a graph  $G$  has the HSCP if every non-trivial half-space (that is a half-space which is different from  $\emptyset$  and from  $V(G)$ ) in this graph is a copoint at each of its neighbors.

Since a median graph is a bipartite weakly median graph, it follows that if  $A$  is a convex set of a median graph  $G$ , then  $|A^{(u)}| = 1$  for each  $u \in N_G(A)$  and  $A^{(u)} \cap A^{(v)} = \emptyset$  for all distinct  $u, v \in N_G(A)$ . Therefore, Proposition 4.3 implies immediately that median graphs have the HSCP.

The HSCP is not a characteristic property of median graphs as is shown by the example of the octahedron  $K_{2,2,2}$  (Fig. 1). This graph is a non-median weakly median graph. Its non-trivial half-spaces are the triples of its adjacent vertices. Hence, by Proposition 4.3, each of these sets is a copoint at each of its neighbors.

**Definition 4.4.** We will say that a graph  $G$  is *semi-octahedral* if, for every triple  $\{u_0, u_1, u_2\}$  of pairwise adjacent vertices of  $G$ , the set  $N_G(u_i) \cap N_G(u_j) - N_G[u_k]$  is non-empty for each triple  $(i, j, k)$  of distinct elements of  $\{0, 1, 2\}$  (Fig. 2). If moreover every edge of  $G$  is contained in a triangle, then we will say that  $G$  is *octahedral*.

In other words a graph  $G$  is *octahedral* if and only if any two adjacent vertices of  $G$  have two non-adjacent common neighbors. In particular, the octahedron is an octahedral weakly median graph and it is clearly the smallest of them. The icosahedron is also an octahedral graph, but it is not weakly modular. Due to the absence of triangles, every



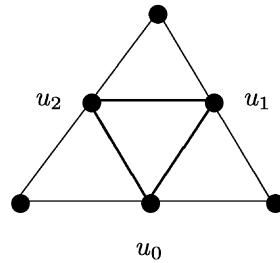


Fig. 2.

bipartite graph, and thus every median graph, is semi-octahedral. Note that the only simplicial vertices (i.e., vertices whose neighborhoods induce simplices) of a connected semi-octahedral graph are the vertices of degree 1.

**Theorem 4.5.** *Let  $G$  be a weakly median graph. The following assertions are equivalent:*

- (i)  $G$  is semi-octahedral.
- (ii)  $G$  is HSCP.
- (iii)  $\text{co}_G(A \cup \{u\}) = N_G[A]$  for every non-trivial half-space  $A$  in  $G$  and each  $u \in N_G(A)$ .

**Proof.** (i)  $\Rightarrow$  (ii): Suppose that  $G$  has not the HSCP. Then there is a non-trivial half-space  $A$  in  $G$  which is not a copoint at some vertex  $u \in N_G(A)$ . By Proposition 4.3, there exists  $v \in N_G(u) \cap N_G(A)$  such that  $A^{(u)} \subseteq A^{(v)}$ . Let  $w \in A^{(u)}$ . Then the vertices  $u, v, w$  are pairwise adjacent. Let  $x \in N_G(u) \cap N_G(w) - \{v\}$ . If  $x \in A$ , then  $x \in A^{(u)}$  and thus  $x \in A^{(v)}$ . If  $x \notin A$ , then  $x$  and  $v$  are adjacent by the compactness of  $V(G) - A$  and the fact that  $\langle v, w, x \rangle$  is a  $(v, x)$ -path with  $w \in A$ . Whence,  $x \in N_G(v)$  in both cases. Therefore,  $N_G(u) \cap N_G(w) \subseteq N_G[v]$ , which proves that  $G$  is not semi-octahedral.

(ii)  $\Rightarrow$  (i): Suppose that there are three pairwise adjacent vertices  $u, v, w$  such that  $N_G(u) \cap N_G(v) \subseteq N_G[w]$ . Since  $\{v, w\}$  is convex, there exists a copoint  $A$  at  $u$  which contains the set  $\{v, w\}$ . Then  $B := V(G) - A$  is a non-trivial half-space. Let  $x \in B^{(v)}$ . Then either  $x$  and  $u$  coincide or are adjacent. Hence  $x \in B^{(w)}$  by the hypothesis, and thus  $B$  is not a copoint at  $v$  by Proposition 4.3. Therefore,  $G$  has not the HSCP.

(ii)  $\Rightarrow$  (iii): Suppose that  $G$  is HSCP, and let  $A$  be a non-trivial half-space in  $G$  and  $u \in N_G(A)$ . Then  $A$  is a copoint at  $u$ . Hence, by Lemma 4.2,  $N_G[A] \subseteq \text{co}_G(A \cup \{u\}) \subseteq N_G[A]$ . Therefore,  $\text{co}_G(A \cup \{u\}) = N_G[A]$ .

(iii)  $\Rightarrow$  (ii) is obvious.  $\square$

**Corollary 4.6.** *For every non-trivial half-space  $A$  in a semi-octahedral weakly median graph  $G$  we have the following two equivalent properties:*

- (i)  $N_G[A]$  is convex.
- (ii)  $N_G(A) \cup N_G(V(G) - A)$  is convex.

**Proof.** (i) is a consequence of Theorem 4.5.

(i)  $\Rightarrow$  (ii): Suppose that  $N_G[A]$  is convex. The set  $V(G) - A$  is also a half-space. Therefore,  $N_G[V(G) - A]$  is convex by (i). Hence  $N_G(A) \cup N_G(V(G) - A) = N_G[A] \cap N_G[V(G) - A]$  is convex too.

(ii)  $\Rightarrow$  (i): Suppose that  $N_G[A]$  is not convex. Due to the connectedness of  $N_G[A]$  and [11, Theorem 7], it follows that there exist two vertices  $x, y$  at distance 2 in the subgraph of  $G$  induced by  $N_G[A]$  and a common neighbor  $u$  of these two vertices which does not belong to  $N_G[A]$ . Then  $x, y$  must belong to  $N_G(A)$ . Hence, their distance in the subgraph of  $G$  induced by  $N_G(A) \cup N_G(V(G) - A)$  is also 2 and  $u \notin N_G(A) \cup N_G(V(G) - A)$ , which proves that  $N_G(A) \cup N_G(V(G) - A)$  is not convex by [11, Theorem 7].  $\square$

**Lemma 4.7.** *A Cartesian product of two graphs  $G_1$  and  $G_2$  is octahedral (resp. semi-octahedral) if and only if  $G_1$  and  $G_2$  are octahedral (resp. semi-octahedral).*

This is clear because, in a Cartesian product, a triangle always has a triangle as a projection onto one of the factors of the product.

An isometric subgraph of an octahedral (resp. semi-octahedral) weakly median graph  $G$  (which is obviously a weakly median subgraph of  $G$ ) is octahedral (resp. semi-octahedral) if it is  $\Delta$ -closed. Consequently, each gated subgraph of  $G$  is an octahedral (resp. semi-octahedral) weakly median subgraph of  $G$ . Because every gated subgraph is  $\Delta$ -closed, it follows that if a graph  $G$  is a gated amalgam of two weakly median graphs  $G_1$  and  $G_2$ , then each triangle of  $G$  is a triangle of  $G_1$  or of  $G_2$ . Whence the following result:

**Lemma 4.8.** *A gated amalgam of two weakly median graphs  $G_1$  and  $G_2$  is octahedral (resp. semi-octahedral) if and only if  $G_1$  and  $G_2$  are octahedral (resp. semi-octahedral).*

**Theorem 4.9.** *The non-trivial elementary octahedral (resp. semi-octahedral) weakly median graphs are the hyperoctahedra and the octahedral two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graphs (resp.  $K_2$ , the hyperoctahedra and the octahedral two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graphs). The latter bridged graphs are exactly the graphs which can be realized as countably infinite two-connected plane graphs such that all inner faces are triangles, all inner vertices have degrees greater than 5, at most finitely many vertices lie on the interior of each region of the plane bounded by a cycle, and no edge lies on the boundary of an outer face.*

**Proof.** An elementary octahedral or semi-octahedral weakly median graph must be an elementary weakly median graph. Therefore, we have to check among the elementary weakly median graphs (Theorem 3.4) which are octahedral and which are semi-octahedral. Obviously a  $n$ -wheel is not semi-octahedral.

(a)  $K_2$  is semi-octahedral, and every semi-octahedral elementary subhyperoctahedron with at least three vertices is octahedral since every edge is contained in a triangle.

Let  $G := K_{i_1, i_2, \dots}$  with  $1 \leq i_j \leq 2$  for every  $j$  be an elementary octahedral subhyperoctahedron. We can assume that  $G$  has at least six vertices because the smallest octahedral weakly median graph is the octahedron which has six vertices. Suppose that  $G$  contains a universal vertex  $u_1$ . Clearly  $u_1$  belongs to some triangle  $\{u_1, u_2, u_3\}$ . Since  $u_1$  is adjacent to all vertices of  $G$ ,  $N_G[u_1] = V(G)$ , and thus  $N_G(u_2) \cap N_G(u_3) - N_G[u_1]$  is empty. Hence the graph is not octahedral. Therefore, all  $i_j$ 's are equal to 2. Let  $u_1, u_2, u_3$  be any three pairwise adjacent vertices of  $G$  which are not adjacent to  $u'_1, u'_2, u'_3$ , respectively. For each triple  $(i, j, k)$  of distinct elements of  $\{0, 1, 2\}$ ,  $N_G(u_i) \cap N_G(u_j) - N_G[u_k] = \{u'_k\}$ . Thus, the finite elementary octahedral subhyperoctahedra are the hyperoctahedra  $K_{2,2,2,\dots}$  with at least six vertices.

(b) We will now prove that a finite two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graph is not semi-octahedral. Let  $G$  be such a graph. Since  $G$  is a finite bridged graph, by a result of Anstee and Farber [1, Theorem 2.1], there exists a vertex  $x$  of  $G$  which is dominated by one of its neighbors  $y$ . Because  $G$  is two-connected and  $K_4$ - and  $K_{1,1,3}$ -free, the degree of  $x$  is 2 or 3. If  $x$  has degree 3, then the two neighbors of  $x$  distinct from  $y$  cannot be adjacent since  $G$  is  $K_4$ -free. Therefore in both cases, if  $u \in N_{G-y}(x)$ , then  $y$  is the only common neighbor of  $x$  and  $u$ , which proves that  $G$  is not semi-octahedral.

Therefore, by Theorem 3.4, an octahedral two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graph can be realized as a countably infinite two-connected plane graph such that all inner faces are triangles, all inner vertices have degrees greater than 5, and at most finitely many vertices lie on the interior of each region of the plane bounded by a cycle. Suppose that an edge  $e$  lies on the boundary of an outer face. Since  $G$  is octahedral,  $e$  must be an edge of two distinct triangles. Hence, because  $e$  lies on the boundary of an outer face, the vertex of one of the triangles which is not incident with  $e$  must lie on the interior of the region of the plane bounded by the other triangle, contrary to the fact that all triangles of the drawing of  $G$  constitute inner faces (see part (3d) of the proof of Theorem 3.4).

Conversely, let  $G$  be a countably infinite two-connected plane graph such that all inner faces are triangles, all inner vertices have degrees greater than 5, at most finitely many vertices lie on the interior of each region of the plane bounded by a cycle, and no edge lies on the boundary of an outer face. By Theorem 3.4,  $G$  is a two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graph. Let  $x, y$  be two adjacent vertices of  $G$ . By hypothesis, the edge  $\{x, y\}$  does not lie on the boundary of an outer face. Hence this edge lies on the boundary of two inner faces, that is, of two triangles whose vertices are  $\{u, x, y\}$  and  $\{v, x, y\}$ , respectively. The vertices  $u, v$  cannot be adjacent, since otherwise one of the vertices  $x, y$  would lie on the interior of the region of the plane bounded by the triangle induced by  $u, v$  and the other vertex, contrary to the fact that all triangles of  $G$  constitute inner faces. This proves that  $G$  is octahedral.  $\square$

**Theorem 4.10.** *Every non-trivial finite octahedral (resp. semi-octahedral) weakly median graph is obtained by successive gated amalgamation from Cartesian products of hyperoctahedra (resp.  $K_2$  and hyperoctahedra).*

This is an immediate consequence of Lemmas 4.7 and 4.8, Theorem 4.9 and [2, Theorem 1]. We can note that, if any finite prime semi-octahedral weakly median graphs and more generally any gated amalgam of these graphs are pseudo-median, this is, however, not the case for the infinite ones (take for example the *triangular grid*, that is, the tiling of the plane into equilateral triangles of equal size), and for Cartesian products of (even finite) hyperoctahedra (see [4]).

We will recall some elementary properties of the Cartesian product of graphs. If  $G = G_1 \square G_2$ , then, for all  $x, y \in V(G)$  (see [18]):

$$d_G(x, y) = d_{G_1}(\text{pr}_1(x), \text{pr}_1(y)) + d_{G_2}(\text{pr}_2(x), \text{pr}_2(y)),$$

hence

$$\text{diam}(G) = \text{diam}(G_1) + \text{diam}(G_2).$$

Moreover,  $\text{pr}_i(I_G(x, y)) = I_{G_i}(\text{pr}_i(x), \text{pr}_i(y))$  for  $i = 1, 2$ . Therefore, if a subset  $C$  of  $V(G)$  is convex in  $G$ , then  $\text{pr}_i(C)$  is convex in  $G_i$  for  $i = 1, 2$ . Conversely, if  $C_i$  is a convex set in  $G_i$  for some  $i \in \{1, 2\}$ , then  $\text{pr}_i^{-1}(C_i)$  is convex in  $G$ . Consequently, if  $H_i$  is a half-space in  $G_i$  for some  $i \in \{1, 2\}$ , then  $\text{pr}_i^{-1}(H_i)$  is a half-space in  $G$ .

**Lemma 4.11.** *For each vertex  $x$  of a Cartesian product of  $n$  prime finite semi-octahedral weakly median graphs, there exists a unique vertex  $y$  of  $G$ , called the antipode of  $x$  in  $G$ , such that  $d_G(x, y) = \text{diam}(G)$  and  $I_G(x, y) = V(G)$ .*

The proof by induction on  $n$  is straightforward and is left to the reader.

**Theorem 4.12.** *Let  $G$  be a finite semi-octahedral weakly median graph. The following assertions are equivalent:*

- (i)  $G$  is a Cartesian product of prime semi-octahedral weakly median graphs.
- (ii)  $G$  is regular.
- (iii) The non-trivial half-spaces in  $G$  are non-comparable with respect to inclusion (i.e., if  $H_1$  and  $H_2$  are half-spaces in  $G$  with  $H_1 \subseteq H_2$ , then  $H_1 = H_2$ ).

We recall that the *depth* of a convex structure is the largest possible length of a chain of non-trivial half-spaces (see [5]). In the following, by the depth of a graph we will mean the depth of its geodesic convexity. Condition (iii) of the preceding theorem is then equivalent to the following condition:

- (iii')  $G$  has depth 1.

**Proof.** (i)  $\Leftrightarrow$  (ii) is a consequence of Theorem 4.9 and of a result of Brešar [6, Theorem 3.2].

(i)  $\Rightarrow$  (iii): Assume that  $G$  is a Cartesian product of  $n$  prime semi-octahedral weakly median graphs. We will prove by induction on  $n$  that  $G$  satisfies (iii). Let  $n = 1$ . Then  $G$  is a prime semi-octahedral weakly median graph, that is  $K_2$  or a hyperoctahedron. The result is trivial if  $G = K_2$ . Suppose that  $G$  is a hyperoctahedron  $K_{2,2,\dots,2}$  of order  $2k$ . If  $x, y$  are two non-adjacent vertices of  $G$ , then  $\text{co}_G(x, y) = V(G)$ . Hence, the non-trivial half-spaces of  $G$  are the vertex sets of two complementary simplices in  $G$  of order  $k$ . Let  $n \geq 1$ . Suppose that the result holds for any Cartesian product of  $n$  prime semi-octahedral weakly median graphs, and let  $G$  be a Cartesian product of  $n + 1$  prime semi-octahedral weakly median graphs. Then  $G = G_1 \square G_2$  where  $G_1$  is prime and  $G_2$  is a Cartesian product of  $n$  prime semi-octahedral weakly median graphs. Let  $H$  be a non-trivial half-space in  $G$ .

**Claim.** *There exists  $i \in \{1, 2\}$  such that  $\text{pr}_i(H) \cap \text{pr}_i(V(G) - H) = \emptyset$  and thus  $\text{pr}_i(H)$  is a non-trivial half-space in  $G_i$ , while  $\text{pr}_{3-i}(H) = \text{pr}_{3-i}(V(G) - H) = V(G_{3-i})$ .*

Let  $i \in \{1, 2\}$ . Then  $\text{pr}_i(H)$  and  $\text{pr}_i(V(G) - H)$  are non-empty convex sets in  $G_i$ . Clearly  $V(G_i) - \text{pr}_i(H) \subseteq \text{pr}_i(V(G) - H)$  and  $V(G_i) - \text{pr}_i(V(G) - H) \subseteq \text{pr}_i(H)$ . Suppose that  $\text{pr}_1(H)$  is not a non-trivial half-space. Then

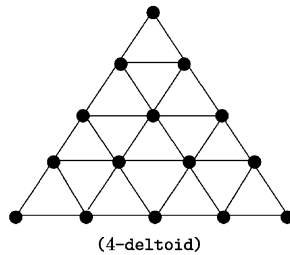


Fig. 3. (4-deltoid).

$\text{pr}_1(H) \cap \text{pr}_1(V(G) - H) \neq \emptyset$ . Let  $x$  be an element of this intersection, and let  $y$  be its antipode in  $G_1$ . Suppose  $\text{pr}_1(H) \neq V(G_1)$ . Then  $y \in V(G_1) - \text{pr}_1(H)$  because  $\text{co}_{G_1}(x, y) = V(G_1)$ . Hence  $x, y \in \text{pr}_1(V(G) - H)$ , which proves that  $\text{pr}_1(V(G) - H) = V(G_1)$ . Let  $x_1 \in \text{pr}_1(H)$ . Then there are  $x_2, y_2 \in V(G_2)$  such that  $(x_1, x_2) \in H$  and  $(x_1, y_2) \notin H$ . Because  $\text{pr}_1(H) \neq V(G_1)$ , there is a vertex  $y_1 \in V(G_1) - \text{pr}_1(H)$ . Since  $\text{pr}_1(V(G) - H) = V(G_1)$ ,  $(y_1, x_2) \notin H$ . Then  $(x_1, x_2) \in I_G((x_1, y_2), (y_1, x_2)) \subseteq V(G) - H$ , contrary to the fact that  $(x_1, x_2) \in H$  by hypothesis. Therefore,  $\text{pr}_1(H) = \text{pr}_1(V(G) - H) = V(G_1)$ .

Suppose now that  $\text{pr}_2(H)$  is not a non-trivial half-space in  $G_2$ . Then, as before, we would have  $\text{pr}_2(H) = \text{pr}_2(V(G) - H) = V(G_2)$ . Let  $x_1, x_2, y_2$  be defined as above. Because  $\text{pr}_2(V(G) - H) = V(G_2)$ , there would exist  $z_1 \in V(G_1)$  such that  $(z_1, x_2) \notin H$ . It would follow that  $(x_1, x_2) \in I_G((x_1, y_2), (z_1, x_2))$ , contrary to the fact that  $(x_1, x_2) \in H$  by hypothesis. Therefore,  $\text{pr}_2(H)$  is a non-trivial half-space in  $G_2$ . Hence  $\text{pr}_2(V(G) - H) = V(G_2) - \text{pr}_2(H)$ . This proves the claim.

Let  $H$  and  $H'$  be two non-trivial half-spaces in  $G$  with  $H \subseteq H'$ . By the preceding claim we can assume that  $\text{pr}_2(H)$  is a non-trivial half-space in  $G_2$ , while  $\text{pr}_1(H) = \text{pr}_1(V(G) - H) = V(G_1)$ . Therefore, since  $\text{pr}_1(H) \subseteq \text{pr}_1(H')$ , it follows that  $\text{pr}_1(H') = \text{pr}_1(V(G) - H') = V(G_1)$ , and thus that  $\text{pr}_2(H')$  is a non-trivial half-space in  $G_2$ . By the induction hypothesis  $\text{pr}_2(H) = \text{pr}_2(H')$ . Then  $\text{pr}_2^{-1}(\text{pr}_2(H))$  and  $\text{pr}_2^{-1}(\text{pr}_2(V(G_2) - \text{pr}_2(H)))$  are complementary half-spaces of  $G$  which contains  $H'$  and  $V(G) - H$ , respectively. It follows that  $H' = \text{pr}_2^{-1}(\text{pr}_2(H)) = H$ .

(iii)  $\Rightarrow$  (i): Suppose that  $G$  is not a Cartesian product of prime weakly median graphs. By Theorem 4.10,  $G$  is a proper gated amalgam of two weakly median graphs  $G_1$  and  $G_2$ . Let  $x_1 \in V(G_1 - G_2)$  and  $x_2 \in V(G_2 - G_1)$ . For  $i = 0, 1$  let  $C_i$  be a copoint at  $x_i$  in  $G$  containing  $V(G_1 \cap G_2)$ . These copoints are non-trivial half-spaces in  $G$ . Then  $V(G) - C_2$  is also a non-trivial half-space in  $G$  with  $V(G) - C_2 \subseteq V(G_2 - G_1) \subset V(G_2) \subseteq C_1$ . Therefore,  $G$  does not satisfy (iii).  $\square$

We obtain immediately:

**Corollary 4.13.** *Let  $G$  be a finite median graph. The following assertions are equivalent:*

- (i)  $G$  is a hypercube.
- (ii)  $G$  is regular.
- (iii) The non-trivial half-spaces in  $G$  are non-comparable with respect to inclusion.

The equivalence (i)  $\Leftrightarrow$  (ii) was already proved by Mulder [21, Corollary 4], and the equivalence (i)  $\Leftrightarrow$  (iii) by Isbell [19] and van de Vel [24].

Note that Theorem 4.12 does not hold if we consider any finite weakly median graph. In fact even a prime weakly median graph can have a depth greater than 1. More precisely, for every  $n \geq 1$  there exists a prime weakly median graph whose depth is  $n$ . For example, the  $n$ -deltoid (see Fig. 3), which is a two-connected  $K_4$ - and  $K_{1,1,3}$ -free bridged graph, has depth  $n$ .

By Theorem 3.5, every (semi-)octahedral weakly median graph is a retract of a Cartesian product of elementary (semi-)octahedral weakly median graphs, but, contrary to the other particular weakly median graphs such as median graphs, quasi-median graphs and pseudo-median graphs, any retract of a (semi-)octahedral weakly median graph is

not necessarily (semi-)octahedral because any triangle of a (semi-)octahedral weakly median graph is a retract of this graph, and  $K_3$  is not semi-octahedral.

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